Stochastic backgrounds of gravitational waves and spherical detectors

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ABSTRACT

The analysis of how a stochastic background of gravitational radiation interacts with a spherical detector is given in detail, which leads to explicit expressions for the system response functions, as well as for the cross-correlation matrix of different readout channels. It is shown that distinctive features of GW induced random detector excitations, relative to locally generated noise, are in practice insufficient to separate the signal from the noise by means of a *single* sphere, if prior knowledge on the GW spectral density is nil. The situation significantly improves when such previous knowledge is available, due to the omnidirectionality and multimode capacities of a spherical GW antenna.

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1 Introduction

Stochastic backgrounds of Gravitational Waves (GWs) are amongst the most interesting sources to be detected by the upcoming new generation of antennas: such kind of gravitational radiation will convey to us e.g. information on the structure of the Universe at its earliest evolutionary stages. This is quite unique to gravitational waves, due to their extremely weak coupling to other forms of matter and radiation, and will therefore greatly enhance our understanding of Cosmology as well as Fundamental Physics. Literature on the possible forms, origins and spectral shapes of various gravitational wave backgrounds is rich —see for example [1] for a detailed review and further references on the subject.

The stochastic nature of gravitational background signals, together with the same stochastic origin of the local noise generated in the detector itself, makes very difficult to tell local noise from actual signal. Therefore, detection strategies to extract that kind of gravitational signals from detector data are based on finding *differential properties* capable to distinguish between both processes.

The standard procedure to identify a stochastic signal is to use *two* (or more) antennas, and cross-correlate their outputs. The concept is that, while local noises in separate detectors are uncorrelated, the stochastic signal is common to both. Signal to noise ratio therefore builds up on the basis of long-term average compensation of local noise fluctuations, and can be seen to develop as $T^{1/4}$, where T is the integration time [2]. Specific strategies for GW background detection have been considered for two bar detectors [3], two interferometers [4], a bar and an interferometer [5] and two spherical detectors [6].

A spherical GW detector is a *multimode* device which, in a number of senses, behaves like an array of bars [7, 8, 9]. This suggests that one might try and take advantage of such multimode capacity to perform a search for a GW background by suitably cross-correlating the different detector readouts. A procedure like this would therefore enable the determination of the GW background spectral density with a *single* detector —a significant advantage to rid the usual method (sketched above) of inherent uncertainties bound to the fact that no two real detectors are exactly identical.

One might *a priori* expect in this direction that specific *signal correlation patterns* happen between pairs of sensor readouts which be *not* shared by the local sources of noise,

thereby making possible to filter the latter out. This paper is concerned with the analysis of the possibilities offered by the sphere, as a multimode device, for the detection of a stochastic background of GWs, defined by a *spectral density* function.

As we shall see, though, local noise and GW background random amplitudes actually show no distinctive correlation pattern in the output channels, except for the obvious fact that GWs are only seen in quadrupole (perhaps also monopole) antenna modes [10]. So, in the absence of some kind of previous information on the functional form of the GW spectrum, the above expectations fail. If, on the other hand, spectral information is available ahead of time (*templates*) then the sphere has a very good performance in its frequency band of sensitivity. This can improve efficiency up to an order of magnitude in energy compared with single readout GW detectors (bars and interferometers), due to the sphere being both *omnidirectional* and *multimode*, as we shall see.

The outline of the article will be the following. In section 2 we set up definitions and notation for the stochastic background of GWs, and in section 3 we derive the sphere's response to this signal. In section 4 we present the correlation functions between the sphere's responses induced by the gravitational signal at two points on its surface, where motion sensors will be attached. Section 5 is dedicated to calculate the same correlation function presented in section 4 but in this case induced by local noise, and discusses, in the light of these results, various possible strategies to retrieve information about the incoming gravitational signal spectrum with a *single* spherical detector. We close with a summary of conclusions in section 6, and also include a brief mathematical appendix.

2 The stochastic signal

We give in this section a few definitions and notation for a stochastic background of gravitational radiation, and its characterization through the spectral density $S_h(\omega)$.

A background of gravitational waves can be expressed as a linear superposition of plane waves coming from all directions and with all possible polarizations,

$$h_{ij}(t,\mathbf{x}) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int d^2\Omega\; e^{-i\omega(t-\mathbf{\Omega}\cdot\mathbf{x})} \int_0^{2\pi} d\psi\; \sum_A \tilde{h}_A(\omega;\mathbf{\Omega},\psi)\; G_{ij}^A(\mathbf{\Omega},\psi) \tag{1}$$

where $\boldsymbol{\Omega}$ is a unit vector pointing to a generic location, ψ is a rotation angle around this vector, $\tilde{h}_A(\omega;\boldsymbol{\Omega},\psi)$ are the wave amplitudes and $G_{ij}^A(\boldsymbol{\Omega},\psi)$ the polarization matrices. General Relativity predicts only *two transverse* polarization modes. Nevertheless, another four degrees of freedom must generally be allowed for. We thus have altogether *six* independent 3×3 symmetric polarization matrices. If we choose a right-handed triad of orthonormal vectors $\boldsymbol{m}$, $\boldsymbol{n}$ and $\boldsymbol{\Omega}$, where $\boldsymbol{\Omega}$ is aligned with the propagation direction, a suitable parametrization for the G-matrices is the following:

$$\begin{aligned}
G_{ij}^{M}(\boldsymbol{\Omega},\psi) &= \sqrt{\frac{2}{3}}\ \delta_{ij} \quad , \quad & G_{ij}^{L_0}(\boldsymbol{\Omega},\psi) &= \sqrt{\frac{1}{3}}\ (3\Omega_i\Omega_j-\delta_{ij})\\
G_{ij}^{L_m}(\boldsymbol{\Omega},\psi) &= m_i\Omega_j+m_j\Omega_i \quad , \quad & G_{ij}^{L_n}(\boldsymbol{\Omega},\psi) &= n_i\Omega_j+n_j\Omega_i\\
G_{ij}^{T_\times}(\boldsymbol{\Omega},\psi) &= m_in_j+n_im_j \quad , \quad & G_{ij}^{T_+}(\boldsymbol{\Omega},\psi) &= m_im_j-n_in_j
\end{aligned} \tag{2}$$

where we give to the index A in $G_{ij}^A(\boldsymbol{\Omega},\psi)$ the following meanings: $A=M$ is the *monopole* component, $A=L_0,L_m,L_n$ are *longitudinal quadrupole* components, and $A=T_+,T_\times$ are the two *transverse quadrupole* components. These matrices satisfy normalization conditions

$$\sum_{ij} G_{ij}^A G_{ji}^B = 2\delta^{AB} \tag{3}$$

If (θ,ϕ,ψ) are Euler angles relating the laboratory frame to the just described wave frame then

$$\begin{aligned}
\boldsymbol{m} &= (\cos\psi\cos\theta\cos\phi-\sin\psi\sin\phi\,,\cos\psi\cos\theta\sin\phi+\sin\psi\cos\phi\,,-\cos\psi\sin\theta)\\
\boldsymbol{n} &= (-\sin\psi\cos\theta\cos\phi-\cos\psi\sin\phi\,,-\sin\psi\cos\theta\sin\phi+\cos\psi\cos\phi\,,\sin\psi\sin\theta)\\
\boldsymbol{\Omega} &= (\sin\theta\cos\phi,\sin\theta\sin\phi,\cos\theta)
\end{aligned} \tag{4}$$

where we can see that a choice for the angles θ,ϕ gives a direction in space and a choice for ψ gives the definition of the two transverse polarization states for a particular direction.

The "*electric*" components $R_{0i0j}(t)$ of the Riemann tensor at the center of mass of the sphere are

$$R_{0i0j}(t) = \frac{1}{2}h_{ij,00}(\mathbf{0},t) = -\int_{-\infty}^{\infty} d\omega\,\omega^2 e^{-i\omega t}\int d^2\Omega\int d\psi\ \sum_{A=0}^{5}\tilde{h}_A(\omega;\boldsymbol{\Omega},\psi)\ G_{ij}^A(\boldsymbol{\Omega},\psi), \tag{5}$$

and these are the only components carrying information on an incoming GW [11].

In this paper we shall be considering an unpolarized stochastic bath of GWs which is also isotropic and stationary. Its statistical properties are therefore encoded in its *power spectrum* function, defined by the following equation:

$$\langle \tilde{h}_A^*(\omega, \mathbf{\Omega}, \psi) \, \tilde{h}_{A'}(\omega', \mathbf{\Omega}', \psi') \rangle = \delta(\omega - \omega') \, \left\{ \frac{1}{4\pi} \delta^2(\mathbf{\Omega}, \mathbf{\Omega}') \, \frac{1}{2\pi} \delta(\psi - \psi') \right\} \left\{ \delta_{AA'} \, \frac{1}{2} S_h(\omega) \right\} \tag{6}$$

where $\langle - \rangle$ stands for *ensemble average*. There are alternative ways to characterize the frequency spectrum, for example, through an energy density per unit logarithmic interval of frequency, or in terms of a characteristic amplitude of the stochastic background. The definitions for such quantities, as well as their mutual relationships can be found in reference [1].

3 The sphere's response

In this section we calculate the sphere's response to the background of gravitational waves given by equation (1). We shall use henceforth the general formalism and notation of reference [10].

We assume that the sphere's response $\mathbf{u}(\mathbf{x}, t)$ to an incoming signal is the solution to the partial differential equation

$$\varrho \frac{\partial^2 \mathbf{u}}{\partial t^2} - \underbrace{\left\{ \mu \, \nabla^2 \mathbf{u} + (\lambda + \mu) \, \nabla(\nabla \cdot \mathbf{u}) \right\}}_{\mathcal{L}\mathbf{u}} = \mathbf{f}(\mathbf{x}, t) \tag{7}$$

with suitable initial and boundary conditions. Here, λ and μ are the material's elastic Lamé coefficients [12], and ϱ is the sphere's density. In the rhs of equation (7), $\mathrm{f}_i(\mathbf{x}, t) = \varrho c^2 R_{0i0j}(t) x_j$ is the *density* of gravitational wave *tidal* forces, ensuing from the *geodesic deviation* equation —see e.g. [13]. The latter term can be split up into its monopole and quadrupole components according to [10]

$$\mathbf{f}(\mathbf{x}, t) = \mathbf{f}^{(00)}(\mathbf{x}) \, g^{(00)}(t) \; + \sum_{m=-2}^{2} \mathbf{f}^{(2m)}(\mathbf{x}) \, g^{(2m)}(t) \tag{8}$$

with

$$
\begin{aligned}
\mathrm{f}_i^{(00)}(\mathbf{x}) &= \varrho\, E_{ij}^{(00)}\, x_j \quad , \quad g^{(00)}(t) = \frac{4\pi}{3}\, E_{ij}^{*(00)}\, R_{0i0j}(t)\, c^2 \\
\mathrm{f}_i^{(2m)}(\mathbf{x}) &= \varrho\, E_{ij}^{(2m)}\, x_j \quad , \quad g^{(2m)}(t) = \frac{8\pi}{15}\, E_{ij}^{*(2m)}\, R_{0i0j}(t)\, c^2 \qquad (m=-2,\ldots,2)
\end{aligned}
\tag{9}
$$

The definition for the matrices $E_{ij}^{(00)}$ and $E_{ij}^{(2m)}$ can be found in appendix A. The generic response function $\mathbf{u}(\mathbf{x},t)$ can be expressed as an orthogonal series expansion, which only involves quadrupole and monopole terms:

$$
\mathbf{u}(\mathbf{x},t) = \sum_{n=1}^{\infty} \frac{a_{n0}}{\omega_{n0}}\, \mathbf{u}_{n00}(\mathbf{x})\, g_{n0}^{(00)}(t) \;+\; \sum_{n=1}^{\infty} \frac{a_{n2}}{\omega_{n2}} \left[\sum_{m=-2}^{2} \mathbf{u}_{n2m}(\mathbf{x})\, g_{n2}^{(2m)}(t) \right] \tag{10}
$$

where $\mathbf{u}_{nlm}$ are *spheroidal* wavefunctions corresponding to the $(2l+1)$-degenerate eigenfrequency ω_{nl}, i.e.,

$$
\mathcal{L}\mathbf{u}_{nlm}(\mathbf{x}) = -\omega_{nl}^2 \varrho \mathbf{u}_{nlm}(\mathbf{x}) \tag{11}
$$

with $\mathcal{L}$ the differential operator defined in equation (7); a_{n0} and a_{n2} are projection coefficients —see [14] for details on definitions and values of these quantities.

Finally, $g_{nl}^{(lm)}(t)$ are *convolution products* between the driving terms $g^{(lm)}(t)$ and the corresponding eigenmode oscillation function —in this case represented by an ideally non-dissipating, purely sinusoidal vibration:

$$
g_{nl}^{(lm)}(t) \equiv \int_0^t dt'\, g^{(lm)}(t')\, \sin\{\omega_{nl}(t-t')\} \qquad (l=0,2\ ;\ m=-l,...,l) \tag{12}
$$

We shall later relax this ideal behavior hypothesis in order to cope with the stationary state the system will eventually reach under the action of long term random excitations.

We next calculate the sphere's response to the background of gravitational waves given by equation (1). The only quantities we have to evaluate are the functions $g_{nl}^{(lm)}(t)$ using equations (12) and (9), with the components of the Riemann tensor given by equation (5). The following is readily found:

$$
g_{nl}^{(lm)}(t) = -\int_{-\infty}^{\infty} d\omega\, \omega^2\, T(t;\omega,\omega_{nl}) \int d^2\Omega\, d\psi\, \sum_A \tilde{h}_A(\omega;\mathbf{\Omega},\psi)\, G_{ij}^A(\mathbf{\Omega},\psi)\, E_{ij}^{*(lm)} \tag{13}
$$

where

$$T(t;\omega,\omega_{nl}) \equiv \int_0^t dt'\, e^{-i\omega t'}\, \sin\{\omega_{nl}(t-t')\} \tag{14}$$

is a function strongly peaked at $\omega = \pm\omega_{nl}$ for large values of t. The sphere's response $\mathbf{u}(\mathbf{x},t)$ is then given by equation (10) with $g_{nl}^{(lm)}(t)$ given by equation (13).

In actual practice, the sphere's motions are sensed at a number of different positions on its surface, where *motion sensors* are attached. We consider devices which are only sensitive to radial displacements, and attached to the sphere's surface at positions

$$\mathbf{x}_a = R\,\mathbf{n}_a\ , \qquad a=1,\ldots,J \tag{15}$$

where R is the radius of the sphere, and $\mathbf{n}_a$ a unit vector pointing outward at the a-th position. There will consequently be J readout channels with output displacements

$$u_a(t) \equiv \mathbf{n}_a\cdot\mathbf{u}(\mathbf{x}_a,t)\ , \qquad a=1,\ldots,J \tag{16}$$

If equation (10) is now used, these are seen to be

$$\begin{aligned} u_a^{GW}(t) \;\; &= \;\; \sum_{n=1}^{\infty}\frac{a_{n0}}{\omega_{n0}}\,A_{n0}(R)\,Y_{00}(\mathbf{n}_a)\,g_{n0}^{(00)}(t) \;\; + \\ &+ \;\; \sum_{n=1}^{\infty}\frac{a_{n2}}{\omega_{n2}}\,A_{n2}(R)\left[\sum_{m=-2}^{2}Y_{2m}(\mathbf{n}_a)\,g_{n2}^{(2m)}(t)\right]\ , \quad a=1,\ldots,J \end{aligned} \tag{17}$$

where Y_{lm} are spherical harmonics, and $A_{nl}(R)$ is a radial function coefficient in the spheroidal wavefunction $\mathbf{u}_{nlm}(\mathbf{x})$; this one depends on whether we are dealing with a solid or a hollow sphere —see [10] or [15] for each case, respectively.

Because the time dependent terms $g_{n2}^{(2m)}(t)$ in (17) are *random* in nature, the readouts $u_a(t)$ are consequently random, too. In addition, the linear character of the relationship between both ensures that certain properties of the driving terms, such as e.g. gaussianity, directly carry over to the outputs. We now propose to investigate the relevant statistical properties of the latter.

4 Cross-correlation functions

As already stated in the Introduction, an essential ingredient in our analysis are the *cross-correlations* between different output channels. In order to make them useful, however, we first need to establish their *theoretical* structure, i.e., to assess their behavior under ideal conditions of infinite integration times, and relate them to the unique characteristic function $S_h(\omega)$ —the signal spectral density.

Correlations between the outputs at the readout channels are naturally defined by

$$R_{ab}(t,\tau) \equiv \langle u_a^*(t)\, u_b(t+\tau)\rangle \tag{18}$$

where $\langle - \rangle$ stands for ensemble average again. We must now introduce the function $u_a^{GW}(t)$, given by equation (17), into equation (18) to calculate the correlation functions we are looking for. When averages are taken, equation (6) for the ensemble average of the wave amplitudes comes into play. Dirac δ-functions appearing in that equation allow us to easily perform the integrals in $\mathbf{\Omega}'$, ψ' and ω'. The remaining integrals in $\mathbf{\Omega}$, ψ can then be done after evaluating

$$G_{ijkl} \equiv \int \frac{d\psi}{2\pi} \int \frac{d^2\Omega}{4\pi} \sum_A e_{ij}^A(\mathbf{\Omega},\psi)\, e_{kl}^A(\mathbf{\Omega},\psi) \tag{19}$$

which yields the following result:

$$G_{ijkl} = \underbrace{\frac{2}{3}\,\delta_{ij}\delta_{kl}}_{monopole} + \underbrace{2N\left[-\frac{1}{15}\,\delta_{ij}\delta_{kl} + \frac{1}{10}\,(\delta_{ik}\delta_{jl}+\delta_{il}\delta_{jk})\right]}_{quadrupole} \tag{20}$$

The index A in equation (19) runs over the polarization modes of the incoming gravitational signal. And the result given in equation (20) takes into account the monopole mode and N quadrupole modes (note that $N \leq 5$, for example $N=2$ in General Relativity, and that all of them have the same contribution to G_{ijkl}).

It is now readily seen that there is no cross-correlation between monopole and quadrupole modes, and that the correlation functions between the outputs induced by the gravitational signal split up as

$$R_{ab}^{GW}(t,\tau) = R_{ab}^{(0)GW}(t,\tau) + R_{ab}^{(2)GW}(t,\tau) \tag{21}$$

where $R_{ab}^{(0)GW}$ and $R_{ab}^{(2)GW}$ are monopole and quadrupole terms, respectively, and have the form

$$\begin{aligned}
R_{ab}^{(0)GW}(t,\tau) &= 2\ \ P_0(\mathbf{\Omega}_a\cdot\mathbf{\Omega}_b)\sum_{n,n'=1}^{\infty}\frac{a_{n0}a_{n'0}\,A_{n0}(R)A_{n'0}(R)}{\omega_{n0}\omega_{n'0}}\,f_{nn'}^{(0)}(t,\tau)\\
R_{ab}^{(2)GW}(t,\tau) &= \frac{4N}{15}\,P_2(\mathbf{\Omega}_a\cdot\mathbf{\Omega}_b)\sum_{n,n'=1}^{\infty}\frac{a_{n2}a_{n'2}\,A_{n2}(R)A_{n'2}(R)}{\omega_{n2}\omega_{n'2}}\,f_{nn'}^{(2)}(t,\tau)
\end{aligned}\tag{22}$$

with P_0 the zero-th Legendre polynomial —which is identically equal to 1 for any value of its argument, but we keep it explicit for the sake of structure clarity—, P_2 is the second Legendre polynomial, and

$$f_{nn'}^{(l)}(t,\tau)\equiv\int_0^\infty\frac{d\omega}{2\pi}\,\omega^4\,Re\left[T^*(t,\omega,\omega_{nl})\,T(t+\tau,\omega,\omega_{n'l})\right]\,S_h(\omega)\ ,\qquad l=0,2 \tag{23}$$

Here we see how the spectral density $S_h(\omega)$ characterizing the incoming signal appears, and its relation to the correlation functions.

In a stationary random process, the second order correlation functions are time-shift invariant (by definition), i.e., if we use the definition (18) then the function R should *only* depend on its second argument, τ. Time-shift invariance has been assumed for the exciting GW bath, as we have characterized it by the spectral density $S_h(\omega)$. Nevertheless, the sphere's response is calculated with the series expansion (10) in which a choice for the state of the sphere at $t=0$ (initial conditions) is assumed which, therefore, breaks that invariance.

The correlation functions we are looking for can actually be obtained from (21) and (22) by taking their limit when t approaches infinity, because in this limit the property of time shift invariance is recovered, as the system looses memory of any particular initial conditions in the remote past.

Memory is however not lost in the kind of ideal, non-dissipative elastic body we have described in the preceding sections. Meaningful results can only be obtained if some kind of dissipation, no matter how large or small, is present in the system. This is therefore the appropriate place to introduce dissipation, and we shall do it in the standard way of

assuming an exponentially decaying amplitude for the oscillation eigenmodes, i.e., we make the replacement

$$\sin(\omega_{nl}\tau) \quad \longrightarrow \quad e^{-\gamma_{nl}\tau}\,\sin(\omega_{nl}\tau) \tag{24}$$

where γ_{nl} is proportional to the inverse of the decay time of the associated mode, which is in all cases of interest much larger than the period of oscillation of that mode, or

$$\gamma_{nl} \ll \omega_{nl} \tag{25}$$

If we express the limits of $R_{ab}^{(0)GW}(t,\tau)$ and $R_{ab}^{(2)GW}(t,\tau)$ by dropping the first argument then the following results are readily obtained:

$$R_{ab}^{(0)GW}(\tau) = \quad 2 \quad P_0(\mathbf{\Omega}_a\cdot\mathbf{\Omega}_b)\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\;\tilde{R}^{(0)}(\omega)\,\frac{1}{2}S_h(\omega)\,e^{i\omega\tau} \tag{26}$$

$$R_{ab}^{(2)GW}(\tau) = \frac{4N}{15}\,P_2(\mathbf{\Omega}_a\cdot\mathbf{\Omega}_b)\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\;\tilde{R}^{(2)}(\omega)\,\frac{1}{2}S_h(\omega)\,e^{i\omega\tau} \tag{27}$$

where

$$\tilde{R}^{(l)}(\omega) = \sum_n\,[a_{nl}A_{nl}(R)]^2\,\omega^4\,|L_{nl}(\omega)|^2 \tag{28}$$

and $L_{nl}(\omega)$ is a Lorentzian curve:

$$L_{nl}(\omega) \equiv \frac{1}{\omega^2-\omega_{nl}^2+2i\gamma_{nl}\,\omega} \tag{29}$$

Equations (26) and (27) have a foreseeable *functional* structure: the correlation functions are Fourier transforms of the spectral density of the driving stochastic signal times the sphere's transfer function $\tilde{R}^{(l)}(\omega)$, and this is a characteristic sum of Lorentzian curves centered on the resonant frequencies, and with appropriate weights. Note also that different harmonics (n,n') appear to be uncorrelated.

4.1 Mode channels

The *algebraic* structure of the correlation function *matrix* requires some further consideration. For $P_2(\mathbf{\Omega}_a\cdot\mathbf{\Omega}_b)$ is a symmetric matrix whose rank[1] is at most 5, which means one can form 5 linear combinations

$$y^{(m)}(t) = \sum_{a=1}^{J} v_a^{(m)}\,u_a(t)\ , \qquad m=-2,\ldots,2 \tag{30}$$

where the coefficient $v_a^{(m)}$ is the a-th component of the (normalized) eigenvector associated to the m-th non-null eigenvalue of $P_2(\mathbf{\Omega}_a\cdot\mathbf{\Omega}_b)$, ζ_m^2, say, which is always positive.

If we now define a new correlation function matrix

$$\mathcal{R}_{mm'}^{GW}(\tau) \equiv \langle y^{(m)*}(t)\,y^{(m')}(t+\tau)\rangle\ , \qquad m,m'=-2,\ldots,2 \tag{31}$$

we readily see that

$$\mathcal{R}_{mm'}^{GW}(\tau) = \frac{4N}{15}\,\zeta_m^2\,\delta_{mm'}\,\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\;\tilde{R}^{(2)}(\omega)\,\frac{1}{2}S_h(\omega)\,e^{i\omega\tau} \tag{32}$$

Thus the 5 quadrupole channels $y^{(m)}(t)$ are *uncorrelated* with one another. They are therefore particularly well suited for improved estimation of the *spectral density* $S_h(\omega)$, since the latter is common to all of them. One may recall here that $y^{(m)}(t)$ are actually *mode channels* for the well known *TIGA* [16] and *PHC* [17] sensor distributions, in which cases

$$v_a^{(m)} \propto Y_{2m}(\mathbf{\Omega}_a) \quad \text{for } \textit{TIGA} \text{ and } \textit{PHC} \tag{33}$$

Note that the fact that the $y^{(m)}(t)$ are uncorrelated is independent of whether the motion sensors are resonant or not, so it applies to a system like the recently proposed *dual sphere* [18], too. Note also that sensor geometries such as *TIGA* and *PHC*, for which the $y^{(m)}(t)$ are *mode channels* —i.e., quantitites whose frequency spectrum $\tilde{y}^{(m)}(\omega)$ is directly proportional to that of the GW quadrupole amplitudes $\tilde{g}^{(2m)}(\omega)$—, allow for a very clear

[1] Actually, $\mathsf{rank}\{P_l(\mathbf{\Omega}_a\cdot\mathbf{\Omega}_b)\} = 2l+1$, provided no two of the J vectors $\mathbf{\Omega}_a$ are parallel, and provided of course that $J\geq 2l+1$ —see [14] for full details.

and direct physical interpretation of the correlation coefficients (32). They are therefore also preferred from the specific point of view of the signals considered in this paper.

Summing up, the main result so far is that cross-correlations between different output channels of a spherical GW detector, whether direct sensor readouts or quadrupole (mode) channels, possess a specific *pattern*, which is given by equations (27) or (32), respectively. The problem is of course that a random signal must be told from also random *local noise*; we thus review next the relevant properties of the latter.

5 Local noise and detection strategies

We study in this section the *multipole* characteristics of the correlation functions between the outputs at two points of the sphere induced by the local noise.

The actual readouts of the spherical antenna are in fact superpositions of those induced by the GW and by the local noise, i.e.,

$$u_a(t) = u_a^{GW}(t) + u_a^{LN}(t) \tag{34}$$

We shall generically consider that

$$u_a^{LN}(t) = \sum_{n,l,m} q_{nlm}(t)\, A_{nl}(R)\, Y_{lm}(\mathbf{n}_a) \tag{35}$$

where $q_{nlm}(t)$ are *narrow band* stochastic time series, with correlation times of order $1/\gamma_{nl}$. Because these $q_{nlm}(t)$ are the amplitudes of different *normal modes* of the solid, they are statistically *uncorrelated*, or

$$\langle q_{nlm}^*(t)\, q_{n'l'm'}(t+\tau)\rangle = \delta_{nn'}\,\delta_{ll'}\,\delta_{mm'}\,\mathcal{F}_{nl}(\tau) \tag{36}$$

with,

$$\mathcal{F}_{nl}(\tau) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, |L_{nl}(\omega)|^2\, \frac{1}{2} S_{LN}(\omega)\, e^{i\omega\tau} \tag{37}$$

where $L_{nl}(\omega)$ is given by (29), and $S_{LN}(\omega)$ is the input spectral density of local detector noise. We need not go into the details of the specific form of this function here, which is a superposition of thermal and transducer back-action noise —see [18].

The system readout correlation functions of local noise, defined by

$$R_{ab}^{LN}(\tau) \equiv \langle u_a^{LN*}(t)\, u_b^{LN}(t+\tau)\rangle \tag{38}$$

are therefore given by

$$R_{ab}^{LN}(\tau) = \sum_l \, R_{ab}^{(l)LN}(\tau) \tag{39}$$

where

$$R_{ab}^{(l)LN}(\tau) = \frac{2l+1}{4\pi}\, P_l(\mathbf{\Omega}_a \cdot \mathbf{\Omega}_b) \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \;\; \sum_n |A_{nl}(R)|^2\, |L_{nl}(\omega)|^2\, \frac{1}{2}\, S_{LN}(\omega)\; e^{i\omega\tau} \tag{40}$$

These expressions show that local noise correlations split up in a way similar to (21), except of course that we now have contributions from *all* multipoles rather than just monopole and quadrupole modes, characteristic of GW signals. In particular, we can easily assess how do quadrupole channels —as defined in (30)— correlate with one another in a local noise dominated detector:

$$\begin{aligned} \langle y_{LN}^{(m)*}(t)\, y_{LN}^{(m)}(t+\tau)\rangle \;\; &= \\ & \frac{5}{4\pi}\, \zeta_m^2\, \delta_{mm'} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \;\; \sum_n |A_{n2}(R)|^2\, |L_{n2}(\omega)|^2\, \frac{1}{2}\, S_{LN}(\omega)\; e^{i\omega\tau} \\ + \sum_{l\neq 2} \frac{2l+1}{4\pi}\, \mathcal{P}_{mm'}^{(l)} & \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \;\; \sum_n |A_{nl}(R)|^2\, |L_{nl}(\omega)|^2\, \frac{1}{2}\, S_{LN}(\omega)\; e^{i\omega\tau} \end{aligned} \tag{41}$$

where

$$\mathcal{P}_{mm'}^{(l)} = \sum_{a,b} v_a^{(m)*} v_b^{(m')}\, P_l(\mathbf{\Omega}_a \cdot \mathbf{\Omega}_b)\;, \qquad m = -2, \ldots, 2 \tag{42}$$

is generally a non-diagonal matrix[2].

The important conclusion to be drawn from equations (40) and (41) is that monopole and quadrupole correlations follow precisely the *same algebraic pattern*, whether they are

[2] Exceptionally, it is diagonal for certain values of l in certain sensor configurations —see a discussion in [19].

dominated by local noise or by a bath of random GWs. Extra terms, such as those in the last line of (41), do not really provide a distinctive feature of local noise, for they are only significant at frequencies different from those of the quadrupole modes, as determined by the presence of the frequency dependent coefficients $L_{nl}(\omega)$ in each case.

This is a very unfortunate circumstance, indeed, for it in principle renders useless the cross-correlation algorithm between different detector channels as an efficient means of filtering out local noise. Are there expedient alternatives?

5.1 Filtering strategies

There are two interesting possibilities one can try to filter out local noise effects with a *single* multimode detector:

i) Since different normal modes are statistically uncorrelated, one is tempted to consider e.g. the first and second quadrupole modes of a sphere [9, 15, 18] as a pair of independent antennas, then use cross-correlation between them to filter local noise out.

This however cannot possibly work because, even if local noise induced fluctuations are independent in either mode, so are also GW induced fluctuations. And, as we have just seen, there is no distinctive feature as to how noise gets added to the signal at different quadrupole modes. Actually, as seen in equations (31) and (41), there is no way to tell which part in $y^{(m)}(t)$ is due to GWs and which is due to local noise.

ii) The fundamental difference between a bath of random GWs and local noise is that, while the former can only affect *quadrupole* modes (or, at most, also monopole modes), local noise excites *all* modes, instead —not just quadrupole/monopole. Given that, in a realistic GW antenna, mode linewidths are extremely narrow, i.e., energy transfer between different modes is extremely slow, an attractive possibility to use a *single* sphere as a detector of a *continuous* flow of stochastic GWs would be to identify the effect of the latter as a *temperature excess* in the quadrupole modes, relative to other modes —dipole, octupole, etc.

While this may look like a reasonable approach, it is in fact an impractical one, as we can appreciate by the following argument. The temperature of a mode is of course

a measure of the variance of its random oscillations; since these are a *narrow band* stochastic process, their correlation time, $\tau_{\rm corr}$, is quite long, hence a reliable estimate of the variance requires taking data during a suitable number of correlation times for averaging. If $t_{\rm obs}$ is such time then the *relative* error in the temperature estimation is of the order of

$$\frac{\delta T}{T} \simeq \sqrt{\frac{\tau_{\rm corr}}{t_{\rm obs}}} \tag{43}$$

Let us consider a few likely numbers: an SQL detector —such as described e.g. in [18]— should be capable of sensing energy innovations of magnitude kT/Q, where k is Boltzmann's constant, T is the temperature of the mode (e.g. the first quadrupole), and Q its mechanical quality factor. An optimized device should be able to reach $T/Q \sim 10^{-8}\,{\rm K}^{-1}$, for example with $Q=10^7$ and $T=100$ milli-Kelvin. A spherical antenna whose considered resonance happens at 1 kHz, say, thus has a correlation time of $2\pi Q/\omega_{12}=10^4$ seconds. On the other hand, the precision required in the measurement of the mode temperature to match the SQL is $\delta T/T=10^{-7}$ for the assumed temperature of 0.1 K. If we now make use of (43) then the conclusion is that an integration time of about 3×10^{10} years is required to detect such a small temperature excess as the optimized antenna permits...

It can be argued that stochastic background signals will actually be *hotter* than the just described precision limit. However nucleosynthesis constraints, for instance, set bounds on the spectral density of GWs to values below $3\times 10^{-24}\,{\rm Hz}^{-1/2}$ at frequencies in the 1 kHz range [1]. This translates into a GW bath temperature of fractions of a micro-Kelvin, for whose accurate estimation integration times of the order of several 10 000 years would be required. The use of *fast variance estimation* techniques [20] may improve on this by between one and two orders of magnitude, but even so one would still be in the range of 100 to 1000 years of integration time —an absurd figure.

There is also little hope that GW backgrounds originating in astrophysical sources, such as e.g. supernovae [21], have an effective temperature above 10^{-7} K or so [1].

Altogether then the *multimode* capabilities of a spherical GW antenna do not seem to offer the practical possibility to make reliable, *single detector* measurements of the spectral density of a background of stochastic GWs. An exception to this however happens when some *a priori* information on the actual form of the spectral function is known ahead of time (*template*) [22]. If this is the case then *optimum filter* techniques can be used, of which a wealth is available in the literature [23].

Of course this is not specific to *spherical* detectors —in fact any single, unimodal GW antenna (such as bars or interferometers) can make advantageous use of the above techniques. Spheres however are more efficient in two senses: first they are *omnidirectional*, and second they can provide up to *five independent estimates* of the spectral density of the GW background, as follows from equation (32): each *quadrupole channel* $y^{(m)}(t)$ can be used to produce one such independent estimate. On average, one gets a factor of 15/8 because of omnidirectionality, times a factor of 5 for quadrupole channels, which make up for 75/8, one order of magnitude (in energy) better than bars and interferometers. Let us however stress that this obviously applies only within the frequency band of sensitivity of each detector; in the case of a dual sphere, even this is quite competitive, see [18].

6 Conclusion

In this paper we have investigated in detail how a spherical detector interacts with a background of gravitational waves, characterized by a frequency dependent *spectral energy density* function, $S_h(\omega)$. In particular, we have found the generic response functions for a device with an arbitrary number of motion sensors, as well as their correlation matrix, both of the system readouts and for the five quadrupole channels —*mode channels* for certain specific sensor layout geometries.

We have discovered that there is no *distinctive pattern* in that correlation matrix, relative to the local noise induced pattern, except that the latter involves *all* the antenna's oscillation eigenmodes rather than just the monopole/quadrupole harmonics, characteristic of generic GWs. This difference proves however insufficient to efficiently filter the stochastic GWs from random local disturbances.

We are thus led to conclude that, in the absence of some *a priori* knowledge about the

GW spectral density $S_h(\omega)$, the multimode character of a spherical detector does not offer a useful alternative to the traditional cross-correlation between two (or more) detectors as a recipe to get rid of local noise effects. However, if such *a priori* knowledge is available then the spherical detector naturally yields one order of magnitude better performance (in energy) than single readout devices, such as bar and interferometric GW detectors.

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Appendix A

A suitable representation for the matrices $E_{ij}^{(00)}$ and $E_{ij}^{(2m)}$ used in equation (9) is the following:

$$E_{ij}^{(00)} = \left\{\frac{1}{4\pi}\right\}^{\frac{1}{2}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad E_{ij}^{(20)} = \left\{\frac{5}{16\pi}\right\}^{\frac{1}{2}} \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

$$E_{ij}^{(2\pm1)} = \left\{\frac{15}{32\pi}\right\}^{\frac{1}{2}} \begin{pmatrix} 0 & 0 & \mp1 \\ 0 & 0 & -i \\ \mp1 & -i & 0 \end{pmatrix} \qquad E_{ij}^{(2\pm2)} = \left\{\frac{15}{32\pi}\right\}^{\frac{1}{2}} \begin{pmatrix} 1 & \pm i & 0 \\ \pm i & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \tag{44}$$

having the properties:

$$E_{ij}^{(00)}\,\Omega_i\Omega_j = Y_{00}(\mathbf{\Omega}) \qquad , \qquad E_{ij}^{(2m)}\,\Omega_i\Omega_j = Y_{2m}(\mathbf{\Omega}) \tag{45}$$

where $\mathbf{\Omega}$ is a unit vector, and $Y_{lm}(\mathbf{\Omega})$ are spherical harmonics.

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